Assignment 8.

Cauchy's Integral Formula.

This assignment is due Wednesday, March 20 (because of spring break). Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

In all problems below, L denotes a closed rectifiable simple curve, traversed counterclockwise, with interior $I(L)$ and exterior $E(L)$.

(1) Evaluate the integral using Cauchy's Integral Formula:

$$\int_{|z-a|=a} \frac{z}{z^4-1}dz,$$

where $a \in \mathbb{R}, a > 1$.

(2) Evaluate the integral

$$\frac{1}{2\pi i}\int_L \frac{ze^z}{(z-z_0)^3}dz,$$

where $z_0 \in I(L)$. (*Hint:* Use formula for derivative of Cauchy's integral.)

(3) Prove that

$$\int_0^{2\pi} \cos(\cos\theta)\cosh(\sin\theta)d\theta = 2\pi.$$

(*Hint:* Use average value theorem for $f(z) = \cos z$.)

(4) *Cauchy's formula for an unbounded domain.* Let L be a closed rectifiable simple curve, traversed counterclockwise. Let $f(z)$ be a differentiable function on a domain G, where $L \cup E(L)$ is contained in G. (That is, f is differentiable on a neighborhood of L and *outside* of L, but not necessarily inside.) In particular, $f(z)$ is differentiable at infinity, that is $f(1/w)$ is differentiable at $w = 0$. Suppose that

$$\lim_{z\to\infty} f(z) = A.$$

Prove that

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}d\zeta = A, \quad \text{if } z \in I(L),$$

and

$$\frac{1}{2\pi i}\int_L \frac{f(\zeta)}{\zeta - z}d\zeta = -f(z) + A, \quad \text{if } z \in E(L).$$

(*Hint:* Make the substitution $z = 1/w$, $\zeta = 1/\eta$. You can use the theorem about substitution in complex integral, even though the proof wasn't finished in class.)

(5) Let $f(z) = \frac{2013z^{12}-z^3+3102z^2+100}{(z-1)^3(z-2)^4(z-3)^2(z-4)(z-5)^2}$. Evaluate

$$\int_L \frac{f(\zeta)}{\zeta - z_0}d\zeta,$$

if $I(L)$ contains the disc $|z| \leq 5$, and $z_0 \neq 1, 2, 3, 4, 5$. (*Hint:* Use the problem above.)

NOTE that the "finite" Cauchy's formula is also usable, but much messier in this case.